

MATH Bacc

English version

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0 Introduction

The primary focus of this summary is my tendency to procrastinate when it comes to studying other subjects.

The chapters are not presented in chronological sequence, but rather arranged in a manner I considered most coherent in relation to the theoretical structure.

This summary comprises the full breadth of theoretical content from S6 to S7, including the principal definitions. Exercises are deliberately excluded, and only a limited selection of solution methods—such as techniques involving integrals or analytic geometry—are included. Foundational trigonometric and algebraic concepts have likewise been omitted in the interest of brevity and focus.

In places, the treatment of definitions may be somewhat pedantic—more than the Baccalaureate likely requires—but I've included them for the sake of completeness and, admittedly, a certain fondness for mathematical rigour.

Do note the fact that the translation was done with the assistance of ChatGPT, and that notations may slightly differ.

Should you have any questions, comments, or suggestions regarding the content (or math in general) feel free to let me know.

1 Functions

1.1 Function Analysis (Sem-1 S6)

Definition: Function

A function is a relation between two sets where each element of the first set has exactly one image in the second. Symbolically: $f : A \rightarrow B$, with **function rule** $f(x) \in B$ for all $x \in A$.

Terminology:

- Domain: $\text{dom}f$, the set of input values of f (A).
- Range: $\text{ran}f$, the set of output values of f (B).
- Zero: $f(x) = 0 \Rightarrow x$ is a zero of f .
- Intercepts: x -intercepts: $(\dots, 0)$; y -intercepts: $(0, \dots)$.
- $f(x) = f(-x) \Rightarrow f$ is even.
- $f(-x) = -f(x) \Rightarrow f$ is odd (f may be neither).

Behavior Chart:

- Increasing: $\nearrow$
- Decreasing: $\searrow$
- Local extrema: Max/Min

Sign Chart:

- $f(x) > 0 \Rightarrow +$
- $f(x) < 0 \Rightarrow -$
- $f(x) = 0 \Rightarrow 0$

The behavior chart is derived from the first derivative ($f'(x)$), and optionally the second ($f''(x)$). The sign chart requires the zeros of f .

Types of Functions Zeros must be computable, and Even roots must be non-negative; outside this domain, the function is undefined:

- Polynomials: $\sum_{i=0}^n a_i x^i = a_n x^n + \dots + a_0$ (Horner's scheme may be used for finding zeros)
- Rational functions: $\frac{g(x)}{h(x)}$ (Denominator $\neq 0$; zeros of h are **excluded** from the domain — holes or vertical asymptotes)
- Irrational functions: Root functions
- Exponential functions (See Section 1.5)
- Logarithmic functions (See Section 1.5)
- Trigonometric functions: $a \sin(b(x - c)) + d$
 - Amplitude: a
 - Period: $p = \frac{2\pi}{b}$
 - Midline: $y = d$
 - Possible starting point: (c, d)
 - (Sum and Difference Identities might come in handy)

Function Operations:

- $f + g$: Sum of functions
- $f - g$: Difference of functions
- $f \cdot g$: Product of functions
- $\frac{f}{g}$: Quotient of functions ($g(x) \neq 0$)
- f^{-1} : Inverse of f (if f is invertible)
- $f \circ g$: Composition of f and g , $(f \circ g)(x) = f(g(x))$

Transformations: $a \cdot f(b(x-c)) + d$

- $a > 1$: Vertical stretch; $0 < a < 1$: Vertical compression; $a < 0$: Reflection over the x -axis
- $b > 1$: Horizontal compression; $0 < b < 1$: Horizontal stretch; $b < 0$: Reflection over the y -axis
- $c > 0$: Shift to the right; $c < 0$: Shift to the left
- $d > 0$: Shift upward; $d < 0$: Shift downward

1.2 Limits (Sem-1 S6)

Definition: Limits

The limit of $f(x)$ as $x \rightarrow a$ is defined as the value that $f(x)$ approaches as x gets arbitrarily close to a :

$$\lim_{x \rightarrow a} f(x) = L$$

- Right-hand limit: $\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} f(x)$
- Left-hand limit: $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} f(x)$
- If $\lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$ then $\lim_{x \rightarrow a} f(x)$ does not exist.

The calculation rules for limits **without** indeterminate forms are intuitive and follow the usual operations in $\mathbb{R}$. (Including simple rules with ∞ , such as $a \in \mathbb{R} \Rightarrow a + \infty = \infty$.)

Indeterminate forms: $\infty - \infty, \infty \cdot 0, \frac{\infty}{\infty}, \frac{0}{0}$

Techniques/Rules:

- For $\lim_{x \rightarrow \pm\infty}$ of a polynomial, discard all terms except the one with the highest power.
- $\lim_{a \rightarrow 0} \frac{x}{a} = \pm\infty$: the sign depends on the sign chart and should be interpreted from it.
- $\pm \lim_{x \rightarrow a} \frac{0}{0} \vee \frac{\infty}{\infty}$: divide both numerator and denominator by $(x-a)$.
- $\lim_{x \rightarrow \infty} \infty - \infty$: convert into a product or quotient form.
- For irrational expressions, try to simplify in general.

1.3 Derivatives (Sem-2 S6)

Average slope: $\frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$

General formula for tangents: $y - f(a) = f'(a)(x - a)$

Definition: Derivatives

$f'(x)$ is the function that describes the instantaneous rate of change of $f(x)$. In other words, it gives the rate at which the values of $f(x)$ change with respect to x . Formally, the definition is as follows:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \vee f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

From these definitions, we obtain the following formulas (some of which you should be able to prove yourself):

- $\frac{d}{dx} x^n = nx^{n-1}$
- $\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}}$

- $\frac{d}{dx} \sin(x) = \cos(x)$
- $\frac{d}{dx} \cos(x) = -\sin(x)$
- $\frac{d}{dx} \tan(x) = \frac{1}{\cos^2(x)}$

The following rules also apply:

- $(f \pm g)' = f' \pm g'$
- $(r \cdot f)' = r \cdot f' \quad r \in \mathbb{R}, \text{ i.e., } r \text{ is a constant.}$
- $(f \cdot g)' = f' \cdot g + f \cdot g'$ (Product Rule)
- $\left(\frac{f}{g}\right)' = \frac{f' \cdot g - f \cdot g'}{g^2}$ (Quotient Rule)
- $(f \circ g)' = (f(g(x)))' = f'(g(x)) \cdot g'(x)$ (Chain Rule)

1.4 Applications of Limits and Derivatives (Sem-2 S6)

Definition: Asymptote

$$\lim_{x \rightarrow a^+} f(x) = \pm\infty \vee \lim_{x \rightarrow a^-} f(x) = \pm\infty \implies x = a \text{ is an asymptote.}$$

- $f'(a) > 0 \implies f(x)$ is increasing at point a
- $f'(a) = 0 \implies f(x)$ is constant at point a , a is a local extremum or an inflection point
- $f'(a) < 0 \implies f(x)$ is decreasing at point a
- $f''(a) > 0 \implies f(x)$ is concave up at point a
- $f''(a) = 0 \implies f(x)$ has an inflection point or an extremum at a
- $f''(a) < 0 \implies f(x)$ is concave down at point a

Important: In the cases where $f'(a) = 0$ and $f''(a) = 0$, you must examine the values around a using a sign chart to correctly determine what is happening at point a .

It is important to be able to determine the following characteristics for a function:

- Domain and Range
- Zeros and y-intercept
- Parity
- Asymptotes
- Holes
- Sign Chart and Behavior Chart

Asymptotes

- Horizontal Asymptotes (HA) are found by computing $\lim_{x \rightarrow \pm\infty} f(x)$
- Vertical Asymptotes (VA) exist when a zero appears more often in the denominator than in the numerator
- If a factor appears equally or more often in the numerator than in the denominator, there is a hole
- For logarithmic functions, trivial vertical asymptotes also exist

Extrema Problems These can be easily solved by forming a formula and taking the derivative to find the extrema.

Applications in Physics: $a''(t) = v'(t) = s(t)$ (Respectively: Acceleration, velocity, and position.)

1.5 Exponential and Logarithmic Functions (Sem-2 S6)

Existence conditions for logarithms: $\log_b(f(x)) \implies b > 0, b \neq 1, f(x) > 0$

Derivatives:

- $\frac{d}{dx} a^{f(x)} = a^{f(x)} f'(x) \ln(a)$
- $\frac{d}{dx} \log_a(f(x)) = \frac{f'(x)}{\ln(a)f(x)}$

Exponential Equations and Inequalities:

- Same base $\implies$ apply base properties
- Different bases $\implies$ take logarithms of both terms
- Reducible to a polynomial equation $\implies$ substitute with x and solve the polynomial
- **Note:** For inequalities with a base < 1 , the inequality sign must be reversed for equal bases

Logarithmic Equations and Inequalities:

- Same base $\implies$ apply base properties
- Different bases $\implies$ eliminate the logarithms or try to get the same base
- **Note:** For inequalities with a base < 1 , the inequality sign must be reversed for equal bases

1.6 Real Functions (Sem-1 S7)

For limits of the form $\frac{0}{0}$ or $\frac{\pm\infty}{\pm\infty}$, we use L'Hôpital's Rule:

L'Hôpital's Rule

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0 \vee \pm\infty \quad \text{and} \quad g'(x) \neq 0 \implies \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}.$$

For other indeterminate forms, such as $0 \cdot \infty$ or $\infty - \infty$, L'Hôpital's Rule can be applied after rewriting the expression:

$$0 \cdot \infty \implies \begin{cases} \lim_{x \rightarrow a} f(x) = 0 \\ \lim_{x \rightarrow a} g(x) = \infty \end{cases} \implies \lim_{x \rightarrow a} f(x) \cdot g(x) = \lim_{x \rightarrow a} \frac{f(x)}{\frac{1}{g(x)}} = \frac{0}{0} \quad (\text{or vice versa}).$$

In the case of $\infty - \infty \implies$, by writing the terms over a common denominator, you can often reduce it to an applicable form like $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Slant Asymptotes For limits where $\lim_{x \rightarrow \pm\infty} f(x) = \pm\infty$, we can find a slant asymptote using:

$$\lim_{x \rightarrow \pm\infty} (f(x) - (ax + b)) = 0.$$

We use the following *Cauchy* formulas:

Cauchy Formulas for Asymptotes

$$a = \lim_{x \rightarrow \pm\infty} \frac{f(x)}{x}, \quad b = \lim_{x \rightarrow \pm\infty} (f(x) - ax).$$

When applying these formulas, L'Hôpital's Rule may be useful.

Euclidean Division for Rational Functions For a rational function of the form $\frac{f(x)}{g(x)}$, where the degree of $f(x)$ is exactly one higher than that of $g(x)$, *Euclidean division* can be used:

Euclidean Division for Rational Functions

$$f(x) = g(x)q(x) + r \implies \frac{f(x)}{g(x)} = q(x) + \frac{r}{g(x)}.$$

Where: - $q(x)$ is a linear function that defines the slant asymptote. - $r \in \mathbb{R}$ is the remainder of the division.

As a result, $q(x)$ is the equation of the slant asymptote.

1.7 Integrals (Sem-2 S7)

Definition: Definite Integral

The definite integral $\int_a^b f(x) dx$ represents the area under the graph of f over the interval $[a, b]$. It is defined as the limit of a sum of small rectangles (Riemann sums):

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i) \Delta x.$$

The expression dx represents an infinitesimally small width. Combined with $f(x)$ (the height), it forms the area of a narrow rectangle. You are not expected to explain this rigorously, but it's important to understand why we always write dx in an integral.

Fundamental Theorem of Calculus

Let f be a continuous function on $[a, b]$, and define: $F(x) = \int_a^x f(t) dt$. Then $F(x)$ represents the area under the curve of f from a to x . This function F is differentiable, and:

$$F'(x) = f(x)$$

So F is an antiderivative of f . Moreover:

$$\int_a^b f(x) dx = F(b) - F(a)$$

The area described by an integral is oriented: if it lies above the x -axis, the area is positive; if below, the area is negative. This leads to several fundamental properties:

Basic Properties of Definite Integrals

- $\int_a^a f(x) dx = 0$
- $\int_a^b f(x) dx = -\int_b^a f(x) dx$
- $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$
- $\int_a^b kf(x) dx = k \int_a^b f(x) dx$
- If $g(x) \geq f(x)$ on $[a, b]$, then:

$$\int_a^b g(x) dx \geq \int_a^b f(x) dx$$

Definition: Indefinite Integral

The **indefinite integral** of a function $f(x)$ is the collection of all antiderivatives of f . It is written as:

$$\int f(x) dx = F(x) + C, \quad \text{where } F'(x) = f(x).$$

These two rules also apply to indefinite integrals:

Basic Rules for Indefinite Integrals

- $\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$
- $\int k \cdot f(x) dx = k \cdot \int f(x) dx$

This leads to the most fundamental integrals:

Fundamental Integrals

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (\text{for all } n \in \mathbb{R} \setminus \{-1\})$
- $\int \frac{dx}{x} = \ln|x| + C$
- $\int e^x dx = e^x + C$
- $\int a^x dx = \frac{a^x}{\ln a} + C$
- $\int \sin x dx = -\cos x + C$
- $\int \cos x dx = \sin x + C$
- $\int \frac{dx}{\cos^2 x} = \tan x + C$
- $\int \frac{dx}{\sin^2 x} = -\cot x + C$

Using algebraic manipulation and the rules for integration, some integrals can be simplified into sums or differences of simpler ones. For rational functions, polynomial division can often be applied to break the function into easier parts.

Another essential technique is substitution. For linear substitution $u = ax + b$, we have:

$$\int f(ax + b) dx = \frac{1}{a} \int f(u) du$$

More generally, if you choose a substitution $u = g(x)$, then also $du = g'(x) dx$. The integral is rewritten in terms of u , which typically simplifies the process (with a good choice of u).

When using substitution for definite integrals, be sure to change the limits according to your chosen u .

Integration by Parts

$$\int u(x) \cdot v'(x) dx = u(x)v(x) - \int u'(x) \cdot v(x) dx$$

Choose u so that its derivative is simpler, and let $dv = v'(x) dx$ so that the antiderivative is easily found.

Sometimes, integration by parts must be applied twice for cyclic integrals, such as trigonometric or exponential functions, eventually leading to an equation where the original integral can be isolated.

For definite integrals:

$$\int_a^b u(x) \cdot v'(x) dx = [u \cdot v]_a^b - \int_a^b u'(x) \cdot v(x) dx$$

Definition: Improper Integrals

An **improper integral** involves either infinite bounds or an unbounded function (a singularity) within the interval. There are two main types:

- Type 1: One or both bounds are infinite
- Type 2: The function is undefined (singular) within the interval

In both cases, the integral is defined via a limit — if the limit exists.

The first type is evaluated normally, with limits applied at infinity. The second requires splitting the integral at the singularity. For a singularity $t \in [a, b]$, the integral is split as:

$$\int_a^b f(x) dx = \int_a^t f(x) dx + \int_t^b f(x) dx$$

Again, we evaluate each part using limits as $x \rightarrow t$. **It is crucial** to split the integral properly in this case. Failing to do so leads to incorrect results, as the function is not continuous over the entire interval.

Applications of Integrals

- The *true* area under a curve: $\left| \int_a^b f(x) dx \right|$, split at the roots of $f(x)$ if necessary.
- Area between two functions: $\int_a^b (f(x) - g(x)) dx$, assuming $f(x) \geq g(x)$. Again, split at the roots of $f(x) - g(x)$.
- Arc length of a curve: $\int_a^b \sqrt{1 + (f'(x))^2} dx$
- Volume of a solid of revolution around the x -axis: $\int_a^b \pi (f(x))^2 dx$
- Surface area of a solid of revolution: $\int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} dx$

2 Complex Numbers

2.1 Complex Numbers (Sem-1 S6)

Definition: Complex Numbers

- A complex number is a number of the form $a + bi \in \mathbb{C}$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$ because $\sqrt{-1} \notin \mathbb{R}$.
- The conjugate of a complex number z is $\bar{z}$, so if $z = a + bi \implies \bar{z} = a - bi$.
- $a + bi = c + di \implies a = c \wedge b = d$

Addition (+), subtraction (−), multiplication (·), and powers (a^n) are performed just like in $\mathbb{R}$, using the fact that $i^2 = -1$.

Powers of i : $i^{4k} = 1$; $i^{4k+1} = i$; $i^{4k+2} = -1$; $i^{4k+3} = -i$;

For division ($\frac{a}{b}$), we multiply numerator and denominator by the conjugate of the denominator. This also allows us to compute negative powers ($a^{-n} = \frac{1}{a^n}$). The formula for $(a + b)^{-1} = \frac{a - bi}{a^2 + b^2}$.

The square root ($\sqrt{c + di}$) of a complex number $c + di$ is computed by expanding $(a + bi)^2 = c + di$.

In $\mathbb{C}$, we can find all solutions of a quadratic equation, because we can take the square root of a negative number (discriminant, $\sqrt{b^2 - 4ac}$).

Important: A complex number must never have an imaginary component in the denominator in its final form; this must be simplified (analogous to radicals in the denominator).

2.2 Complex Numbers (Sem-1 S7)

Complex numbers can be visualized on a plane (the *Gauss/Argand* plane) with the real part on the x-axis and the imaginary part on the y-axis.

Standard Notations for Complex Numbers

- Cartesian: $a + bi$
- Trigonometric/Polar: $r(\cos \theta + i \sin \theta)$
- Exponential: $re^{i\theta}$

Properties of the modulus and conjugate:

For a complex number $z = a + bi$, the modulus is $|z| = \sqrt{a^2 + b^2}$. The conjugate $\bar{z}$ has the same modulus, so $|\bar{z}| = |z|$. Furthermore, $\sqrt{z \cdot \bar{z}} = |z|$.

It follows that:

$$|z_1 z_2| = |z_1| \cdot |z_2|, \quad \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad |z^n| = |z|^n$$

The trigonometric form is obtained via: $\tan \theta = \frac{b}{a}$ and $r = |z| = \sqrt{a^2 + b^2}$

Computation rules in polar coordinates:

$\forall z_1 = r_1(\cos \theta_1 + i \sin \theta_1), z_2 = r_2(\cos \theta_2 + i \sin \theta_2) \in \mathbb{C} \implies$

- $z_1 \cdot z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$
- $\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$
- $z^n = r^n (\cos(n\theta) + i \sin(n\theta)), \quad n \in \mathbb{Z}$

$$\sqrt[n]{z} = \sqrt[n]{r} \left(\cos \left(\frac{\theta + 360^\circ \cdot k}{n} \right) + i \sin \left(\frac{\theta + 360^\circ \cdot k}{n} \right) \right), \quad k = 0, 1, \dots, n-1$$

The roots of a complex number lie on a regular n -gon with the origin as its center in the complex plane. There are n roots, and by varying k between n values, one obtains all roots. It is also worth noting that the roots for k and $k + n$ are identical.

Standard operations in exponential form ($z_1 \cdot z_2$, $\frac{z_1}{z_2}$, z^n , $\sqrt[n]{z}$) are performed using exponentiation rules.

3 Probability Theory

3.1 Combinatorics - Counting Problems (Sem-1 S6)

Terminology:

- Ω : sample space $\Rightarrow A \subseteq \Omega$: event
- $A \cup B$: union $A \cap B$: intersection $\bar{A}$ or A^C : Complement
- $A \setminus B$: elements in A but not in B

The counting problems we encountered can be categorized into 4 variables:

- Repetition? Yes or No
- Is order important? Yes or No
- p : number of positions
- n : number of choices ($n \geq p$ always holds for cases without repetition)

These four cases give the following formulas, with repetition (H) and order (V) varying ($\binom{n}{k} = \frac{n!}{k!(n-k)!}$):

- H, V: $V_n^p = \frac{n!}{(n-p)!}$
- H, V: $\overline{V}_n^p = n^p$
- H, V: $C_n^p = \frac{n!}{p!(n-p)!} = \binom{n}{p}$
- H, V: $\overline{C}_n^p = C_{n+p-1}^p = \frac{(n+p-1)!}{p!(n-1)!} = \binom{n+p-1}{p}$

The following formula may also be required:

Binomial Theorem

$$(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}$$

3.2 Probability Theory (Sem-1 S6)

The *law of large numbers* states that if you repeat an experiment enough times, the average result will converge to the true (theoretical) mean. You should be able to use and create the following diagrams: Venn diagram, probability tree, and contingency table.

Formulas:

- Probability of A: $P(A) = \frac{\#A}{\#\Omega}$ Number of outcomes in A divided by the total number of outcomes
- $P(\Omega) = 1$
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $P(A \cap B) = P(A) \cdot P(B|A) = P(B) \cdot P(A|B)$
- $P(\bar{B}) = 1 - P(B)$
- $P(A \setminus B) = P(A) - P(A \cap B)$

The conditional probability is defined as: $P(A|B) = \frac{P(B \cap A)}{P(B)}$ (Probability of A given B)

This formula can be rewritten using Bayes' Rule:

Bayes' Rule

$$P(A|B) = \frac{P(A) \cdot P(B|A)}{P(A) \cdot P(B|A) + P(\bar{A}) \cdot P(B|\bar{A})}$$

Two events are independent if one of the following is true:

- $P(A | B) = P(A)$
- $P(B | A) = P(B)$
- $P(A \cap B) = P(A) \cdot P(B)$

4 Analytical Geometry

4.1 Plane Geometry (Sem-2 S6)

Definition: Vector

A **vector** is a directed line segment with both magnitude and direction, denoted as $\vec{v}$ or $\overrightarrow{AB}$.

Basic concepts:

- Norm: $\vec{u} = \begin{pmatrix} x \\ y \end{pmatrix} \implies ||\vec{u}|| = \sqrt{x^2 + y^2}$
- Dot product (scalar product): $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1 \cdot x_2 + y_1 \cdot y_2$
- Midpoint: $M = \left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2} \right)$
- Linear dependence: If $\vec{a} = k \cdot \vec{b}$, then $\vec{a}$ and $\vec{b}$ are linearly dependent. Otherwise, linearly independent.
- A normal vector is perpendicular to a line.

Equations of a Line

- Vector equation: $y = \vec{A} + k \cdot \overrightarrow{AB}$ ($\vec{A}$ is the position vector, $\overrightarrow{AB}$ is the direction vector)
- Parametric form: $\begin{cases} x = x_1 + a \cdot k \\ y = y_1 + b \cdot k \end{cases}$
- Cartesian equation: $ux + vy + w = 0 \iff \frac{x - x_1}{a} = \frac{y - y_1}{b}$

Relative position of two lines:

- **Strictly Parallel:** Direction vectors are linearly dependent, but no common point exists.
- **Coincident:** Direction vectors are linearly dependent, and all points are shared.
- **Intersecting:** Direction vectors are linearly independent.

Angle Between Two Intersecting Lines

$$\cos(\widehat{\vec{u}, \vec{v}}) = \frac{x_1 \cdot x_2 + y_1 \cdot y_2}{\sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2}} = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| \cdot |\vec{v}|}$$

- **If the value is negative:** it corresponds to the cosine of the obtuse angle between the vectors, and vice versa. -
If the value is 0: the lines are perpendicular $\implies x_1x_2 + y_1y_2 = 0$

- The normal vector is easily read off the Cartesian form: for $ux + vy + w = 0$, the normal vector is $\begin{pmatrix} u \\ v \end{pmatrix}$, so the direction vector is $\begin{pmatrix} -v \\ u \end{pmatrix}$

Applications:

- Orthogonal projection of point A onto line l : Find intersection of l with its normal vector through A .
- Distance from a point $M : (a, b)$ to a line $l : ux + vy + w = 0$:

$$d(M, l) = \frac{|ua + vb + w|}{\sqrt{u^2 + v^2}}$$

- Distance between two parallel lines: Pick a point on one line, then use the point-to-line distance formula.

Equation of a **circle**: $(x - a)^2 + (y - b)^2 = r^2$ with center (a, b) and radius r .

4.2 Solid Geometry (Sem-1 and 2 S7)

Distances and Vectors

Concepts are analogous to 2D:

- Distance between two points: $d((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- Midpoint: $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$
- Norm: $||\vec{u}|| = \sqrt{x^2 + y^2 + z^2}$
- Dot product: $\vec{u} \cdot \vec{v} = x_1x_2 + y_1y_2 + z_1z_2$
- Linear dependence: same as in 2D

Lines and Planes

Line Equations

- Vector equation: $\vec{r} = \vec{A} + r \cdot \vec{AB}, \quad r \in \mathbb{R}$
- Parametric form:
$$\begin{cases} x = x_1 + a \cdot k \\ y = y_1 + b \cdot k \\ z = z_1 + c \cdot k \end{cases}$$
- Cartesian form: $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$

Plane Equations

- Vector equation: $\vec{r} = \vec{r}_0 + s\vec{v}_1 + t\vec{v}_2, \quad s, t \in \mathbb{R}$
- Parametric form:
$$\begin{cases} x = x_1 + sa_1 + ta_2 \\ y = y_1 + sb_1 + tb_2 \\ z = z_1 + sc_1 + tc_2 \end{cases}$$
- Cartesian form: $ux + vy + wz + t = 0$

Relative Positioning:

- Two **lines**:
 - Strictly parallel: No common point (Same direction vectors)
 - Coincident: All points common (Same direction vectors)
 - Intersecting: One common point (Different direction vectors)
 - Skew: No common point (Different direction vectors)
- Two **planes**:
 - Parallel: No common point (Same normal vectors)
 - Coincident: All points common (Same normal vectors)
 - Intersecting: Line of intersection exists (Different normal vectors)
- **Line and plane**:
 - Intersecting: One common point
 - Parallel: No common point (Direction vector is orthogonal to plane's normal)
 - Line lies in plane: All points common (Direction vector is orthogonal to normal vector)

Cross Product

- Cross product (result is perpendicular to the plane formed by two vectors):

$$\vec{a} \times \vec{b} = \begin{pmatrix} y_a z_b - z_a y_b \\ z_a x_b - x_a z_b \\ x_a y_b - y_a x_b \end{pmatrix}$$

- Area of parallelogram: $||\vec{a} \times \vec{b}||$
- Area of triangle: $\frac{1}{2} ||\vec{a} \times \vec{b}||$
- Volume of parallelepiped: $|\vec{a} \cdot (\vec{b} \times \vec{c})|$

Applications

- Normal vector of a plane: $ux + vy + wz + t = 0 \Rightarrow \vec{n}(u, v, w)$
- Orthogonal projection of a point onto a plane: Find intersection with the line through the point in the direction of the normal vector.
- Perpendicular plane through a point on a line: Use the direction vector of the line as the normal vector of the plane.

Distances

- Distance from point to plane: $d = \frac{|ux_1 + vy_1 + wz_1 + t|}{\sqrt{u^2 + v^2 + w^2}}$
- Distance between two parallel lines: Use point-to-line distance with a point on one line and the other line.
- Distance between two parallel planes: Analogous to previous method.
- Distance between skew lines: $d(l_1, l_2) = \frac{|\vec{A_1 A_2} \cdot (\vec{u_1} \times \vec{u_2})|}{|\vec{u_1} \times \vec{u_2}|}$

Angles

- Angle between two vectors: $\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| \cdot ||\vec{v}||}$
- Angle between line and plane: Calculate the angle between the direction vector of the line and the plane's normal vector, then take the complement.
- Angle between two planes: Equal to the angle between their normal vectors.

Equation of a sphere: $(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$ with center (a, b, c) and radius r

5 Sequences

5.1 Sequences (Sem-2 S6)

Definition: Sequence

A sequence of numbers in a specific order is called a **sequence**. Each number is a term, denoted as u_n , where the index is the **position number**.

Types of Definitions

- **Direct/explicit:** $u_n = f(n)$ (There is a direct formula for each term)
- **Recursive:** $u_n = f(u_{n-1}, (u_{n-2}, \dots))$ (Depends on the previous term(s))
 - For a recursive definition, always provide the *initial term*. (A recursive definition can also be doubly recursive (or more), such as in the Fibonacci sequence.)

If a sequence is increasing or decreasing, we call it **monotonic**, which gives the following cases ($\forall n \in \mathbb{N}$):

- Increasing: $u_{n+1} \geq u_n$ or $f'(n) \geq 0$
- Strictly increasing: $u_{n+1} > u_n$ or $f'(n) > 0$
- Decreasing: $u_{n+1} \leq u_n$ or $f'(n) \leq 0$
- Strictly decreasing: $u_{n+1} < u_n$ or $f'(n) < 0$

The value of the limit of a sequence can also be described:

- $\lim_{n \rightarrow \infty} u_n = c, \quad c \in \mathbb{R} \implies$ The sequence **converges**
- $\lim_{n \rightarrow \infty} u_n = c, \quad c = \pm\infty \implies$ The sequence **diverges**

We determine the behavior of the sequence by calculating the limit for $n = \infty$ (explicit) or using a graph (recursive, via NumWorks).

5.2 Arithmetic and Geometric Sequences (Sem-1 S7)

Definition: AR and GS

- An **arithmetic sequence** (AR) is a sequence where the difference between two consecutive terms is always constant.
- A **geometric sequence** (GS) is a sequence where the ratio between two consecutive terms is always constant.

Definitions

- AR recursive: $u_n = u_{n-1} + v$ and Initial term
- AR explicit: $u_n = u_1 + (n-1) \cdot v$ In general: $u_n = u_k + (n-k) \cdot v$
- GS recursive: $v_n = v_{n-1} \cdot q$ and Initial term
- GS explicit: $v_n = v_1 \cdot q^{n-1}$ In general: $v_n = v_k \cdot q^{n-k}$

In general, we also have the following formulas:

- AR sum of the first n terms: $n \cdot \frac{u_1 + u_n}{2}$
- GS sum of the first n terms: $v_1 \cdot \frac{1-r^n}{1-r}$

Convergences and divergences:

- AR, and the sum of an AR, always diverge
- GS converges to 0 if and only if $-1 < q < 1$
- GS converges to a constant if and only if $q = 1$
- GS diverges if and only if $|q| > 1$
- GS sum converges if and only if $-1 < q < 1$, and its value is: $\frac{v_1}{1-q}$

6 Statistics (Not math)

6.1 Discrete Probability Models (Sem-2 S6)

Definition: Basic Definitions

- **Probability model:** A mathematical model to predict experiments involving chance.
- **Discrete:** Consists of separate, distinct values, e.g. $\mathbb{N}$.
- **Continuous:** Can take any value within an interval, e.g. $\mathbb{R}$.
- **Random variable:** A variable that takes random values according to a probability distribution.
- The discrete random variable X takes values $x_1, x_2, \dots, x_n$ with probabilities $p_i = P(X = x_i)$, where $p_i \geq 0$ and $\sum_{i=1}^n p_i = 1$.

Essential Concepts in Statistics

Expected value: $E[X] = \mu_X = \sum_{i=1}^n x_i p_i$

Variance: $Var(X) = \sigma_X^2 = E[(X - \mu_X)^2] = \sum_{i=1}^n (x_i - \mu_X)^2 p_i = \sum_{i=1}^n x_i^2 p_i - \mu_X^2$

Standard deviation: $\sigma_X = \sqrt{Var(X)} = \sqrt{\sum_{i=1}^n (x_i - \mu_X)^2 p_i} = \sqrt{\sum_{i=1}^n x_i^2 p_i - \mu_X^2}$

(For variance, the second formula is easier in my opinion.)

Specific Discrete Probability Distributions:

- **Uniform:** $i, j \in \{1, 2, \dots, n\} \Rightarrow x_i = x_j$, meaning all x_i are equally likely.
- **Bernoulli:** Success p , or failure $q = 1 - p$. $x_1 = 1, x_2 = 0 \Rightarrow \mu_X = p, \sigma_X^2 = pq$
- **Binomial distribution:** Repeated Bernoulli: $X \sim (n, p)$ where n is the number of trials, p the success probability. $P(X = x_i) = \binom{n}{i} p^i q^{n-i}$. $\mu_X = np, \sigma_X^2 = npq$
- **Binomial approximation:** A binomial distribution can be approximated by a normal distribution for large samples.

6.2 Bivariate Statistics (Sem-1 S7)

Definition: Bivariate Statistics

Bivariate statistics studies the relationship between two continuous variables, often visualized as a scatter plot. In practice, we use sample data, so the variables are not truly continuous like in continuous probability models, but discretely measured.

In a scatter plot, we can observe the following characteristics of a bivariate continuous probability distribution:

- Upward trend: y generally increases as x increases
- Downward trend: y generally decreases as x increases
- Linear relationship: A relationship between two variables described by a straight line

We define a typical line that fits the scatter plot as closely as possible:

- It goes through $[\bar{x} - 2s_x; \bar{x} + 2s_x]$ and $[\bar{y} - 2s_y; \bar{y} + 2s_y]$ (Two standard deviations from the mean)
- Positive correlation: $(y - \bar{y}) = \frac{s_y}{s_x}(x - \bar{x})$
- Negative correlation: $(y - \bar{y}) = -\frac{s_y}{s_x}(x - \bar{x})$

In linear correlation, the spread often forms an elliptical shape. The length shows the direction of the line, and the width shows the strength of the correlation, which may be:

- Weak
- Moderate
- Strong
- *Perfect*

$$(|r| = 1)$$

The strength can be determined numerically with the *correlation coefficient* r , though interpretation depends on context.

Z-score: $z_x = \frac{x - \bar{x}}{s_x}$ This measures how far a value is from the mean — the larger it is, the further from average.

To analyze a scatter plot generally, we standardize the model by replacing all values with their Z-scores. For positive correlation, the typical line becomes the first bisector ($y = x$), for negative correlation, the second bisector ($y = -x$). The correlation coefficient is calculated for this standardized model:

Definition: Correlation Coefficient r

The **correlation coefficient** measures the strength of linear association between two variables. It shows if changes in one variable are consistently linked to changes in another.

It is calculated as:

$$r = \frac{1}{n-1} \sum_{i=1}^n \left(\frac{x_i - \bar{x}}{s_x} \cdot \frac{y_i - \bar{y}}{s_y} \right) = \frac{1}{n-1} \sum_{i=1}^n (z_{x_i} \cdot z_{y_i})$$

Interpreting the Correlation Coefficient r

The **correlation coefficient** r measures the strength and direction of the linear relationship between two variables. It ranges between -1 and 1:

- $r > 0 \Rightarrow$ Positive correlation, the variables increase together
- $r < 0 \Rightarrow$ Negative correlation, one decreases as the other increases
- The closer $|r|$ is to 1, the stronger the correlation; the closer to 0, the weaker the correlation

What counts as "high", "moderate", or "low" correlation depends on the context. With proper interpretation and context, meaningful conclusions can be drawn.

Definition: Regression Line

The **regression line** is the line that best fits the linear relationship between two variables. It minimizes the sum of the squared vertical distances (residuals) between the data points and the line.

The regression line is used for *interpolation* (predicting within the data range) and *extrapolation* (predicting beyond the data range).

A *residual plot* shows the residuals of the regression and helps identify outliers or incorrect assumptions in the model.

Outliers can heavily influence the correlation coefficient, so always inspect the scatter plot for anomalies. **Correlation** does not imply **causation** — e.g., higher temperatures may correlate with more ice cream consumption, but that doesn't mean temperature causes it. Be mindful of *spurious correlations* and apply common sense. A weak correlation doesn't rule out other types of relationships.

An exponential relationship $y = a \cdot b^x$ can be linearized by taking logarithms:

$$\ln(y) = \ln(a) + x \cdot \ln(b)$$

This gives a linear relationship between $\ln(y)$ and x , which can then be analyzed using a regression line.

6.3 Continuous Probability Models (Almost Mathematics) (Sem-2 S7)

Definition: Continuous Random Variable

A **continuous random variable** is a variable that can take any real value within an interval, according to a specific **probability density function** $f(x)$, where:

$$P(a \leq X \leq b) = \int_a^b f(x) dx, \quad \text{with} \quad \int_{-\infty}^{\infty} f(x) dx = 1.$$

The following formulas apply:

$$E[X], \text{Var}(X), \sigma_x$$

- $E[X] = \int_{-\infty}^{\infty} x f(x) dx$
- $\text{Var}(X) = \int_{-\infty}^{\infty} (x - \mu_x) f(x) dx$
- $\sigma_x = \sqrt{\text{Var}(X)} = \sqrt{\int_{-\infty}^{\infty} (x - \mu_x) f(x) dx}$

And the following calculation rules (the cases for standard deviation follow trivially from $\text{Var}(X)$):

Calculation Rules

- $E[X + a] = E[X] + a$
- $E[aX] = aE[X]$
- $E[X + Y] = E[X] + E[Y]$
- $\text{Var}(X + a) = \text{Var}(X) \quad \sigma_{X+a} = \sigma_X$
- $\text{Var}(aX) = a^2 \cdot \text{Var}(X) \quad \sigma_{aX} = |a| \cdot \sigma_X$
- $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) \quad \sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2}$

Definition: Uniform Continuous Distribution

Defined on the interval $[a, b]$:

$$f(x) = \begin{cases} 0 & (x < a) \\ \frac{1}{b-a} & (a \leq x \leq b) \\ 0 & (x > b) \end{cases}$$

From this, it follows that $E[X] = \frac{a+b}{2}$, $\text{Var}(x) = \frac{(b-a)^2}{12}$, and $\sigma_x = \frac{b-a}{\sqrt{12}}$.

Definition: Normal Distribution

The **normal distribution** is a continuous probability distribution with a bell-shaped, symmetric curve centered around the mean μ and standard deviation σ . The density function is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The mean is equal to the expected value, and the inflection points of the curve are located at a distance of σ from the mean. This is also known as the *Gaussian distribution*.

The 68-95-99.7 Rule

For a normal distribution, it holds (approximately):

- $\mu \pm 1\sigma$: about 68% of the values
- $\mu \pm 2\sigma$: about 95% of the values
- $\mu \pm 3\sigma$: about 99.7% of the values

A normal distribution is denoted by $X \sim N(\mu; \sigma)$, where μ is the mean and σ is the standard deviation. The mean determines the center of the curve, while the standard deviation indicates how "flat" the curve is — the larger σ , the flatter the distribution.

We can also standardize a normal distribution by considering $Z \sim N(0; 1)$, where Z is the *z-score*. Then Z is said to be **standard normally** distributed. This is useful for solving equations where the mean and/or standard deviation are unknown, but probabilities such as $P(X \leq a) = p$ are given. By standardizing, we can find the corresponding *a*-values for Z , thereby reducing the problem to a linear system of equations.

7 Extra

Here you'll find a more formal explanation (**completely unnecessary for the Baccalaureate**) and a video from the University of Toronto that visually illustrates the concept well. It may help build intuition about how an integral works.

Definition: Definite Integral

The area under a continuous function on a closed interval $[a, b]$ can be determined using an integral. We divide the interval $[a, b]$ into n equal subintervals of length $\Delta x = \frac{b-a}{n}$ and define the points

$$x_0 = a, \quad x_1 = a + \Delta x, \quad \dots, \quad x_n = b.$$

For each subinterval $[x_{i-1}, x_i]$, we determine:

- $m_i = \min\{f(x) \mid x \in [x_{i-1}, x_i]\}$ (minimum function value)
- $M_i = \max\{f(x) \mid x \in [x_{i-1}, x_i]\}$ (maximum function value)

The **lower estimate** of the area is given by the sum:

$$s_n = \sum_{i=1}^n m_i \Delta x$$

and the **upper estimate** by:

$$S_n = \sum_{i=1}^n M_i \Delta x.$$

If f is continuous on $[a, b]$, then:

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} S_n = I.$$

We call this common limit the **definite integral** of f over $[a, b]$, and write:

$$\int_a^b f(x) dx = I.$$

The $\int$ symbol is a stylized S , standing for “sum.” The expression dx indicates that the sum is taken over pieces of width Δx .

Here is a short explanation of the [Riemann sums](#) by the University of Toronto.